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On derived categories of differential complexes

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Abstract

This paper is devoted to the comparison of different localized categories of differential complexes. The main result is an explicit equivalence between the category of differential complexes of order one (defined by Herrera and Lieberman) and the category of differential complexes (of any order, defined by M. Saito), both localized with respect to a suitable notion of quasi-isomorphism. Then we prove a similar result for a filtered version of the previous categories (defined respectively by Du Bois and M. Saito), localized with respect to graded-quasi-isomorphisms, thus answering a question posed by M. Saito.

Keywords: Differential complexes; De Rham complex

Introduction

The category of differential complexes appears naturally as a "good" category for the role of image of the classical De Rham functor. We are interested in finding a purely algebraic definition for the image category of the De Rham functor for differential modules which will permit us to develop the formalism of the six Grothendieck operations.

Such a category was first introduced by Herrera–Lieberman in their article in *Inventiones Math.* of 1971 [7]. In that paper they proposed the study of a category $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ of complexes of $\mathscr{O}_X$-modules with differential operators of order one (where X is a smooth algebraic

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or analytic variety over a field A of characteristic zero). They also interpreted $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ as a category of graded modules over a suitable graded ring $\mathscr{C}_X^\bullet$ containing $\Omega_X^\bullet$ as a sub-ring. Using this interpretation they defined the functors $\_ \otimes_{\Omega^\bullet} \_$, $\mathscr{H}om_{\Omega^\bullet}(\_,\_)$, f^* and f_*. Then they defined hyperext functors using suitable injective resolutions and proved a duality theorem in the proper smooth case.

They did not propose in that paper to localize $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ with respect to a multiplicative system as is done in the study of derived categories, although they did introduce a notion of homotopy.

The difficulty in the localization procedure was first pointed out by P. Berthelot in his book of 1974 [1], where he showed that objects of $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ which are quasi-isomorphic as complexes of abelian sheaves, may lead to nonisomorphic hyperext functors. On the other hand, we are forced to localize $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$, if we wish to obtain a triangulated category where a De Rham functor DR, with source some derived category of $\mathscr{D}_X$-modules, can assume its values. By Berthelot's remark we know that the multiplicative system of abelian quasi-isomorphisms is not a good choice.

Different localizations were proposed by Philippe Du Bois who, in [4], introduced filtrations and so obtained the category $DF_1(\mathscr{O}_X, \mathrm{Diff}_X)$, and by Morihiko Saito who, in [10] and [11], defined a new category of complexes $C(\mathscr{O}_X, \mathrm{Diff}_X)$ with differential operators (of any order) which he localized with respect to DR_X^{-1}-quasi-isomorphism (obtaining $D(\mathscr{O}_X, \mathrm{Diff}_X)$) or with respect to filtered quasi-isomorphism (obtaining $DF(\mathscr{O}_X, \mathrm{Diff}_X)$).

Saito's category $C(\mathscr{O}_X, \mathrm{Diff}_X)$ seems to be the best choice because it is equivalent to the derived category of right $\mathscr{D}_X$-modules denoted here by $D(\mathscr{D}_X)^r$ via the functor $\widetilde{\mathrm{DR}}_X$ (see [11] for the definitions of the functor $\widetilde{\mathrm{DR}}_X$). The problem is that in Saito's category an explicit formalism of Grothendieck operations is only partially realized; in fact, for example, there is no f^* functor or internal tensor product. From the Herrera–Lieberman point of view, considering the category $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$ obtained by localizing $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ with respect to DR_X^{-1}-quasi-isomorphism, we obtain a category wherein the De Rham functor takes its image and where the Grothendieck operations are easier and more complete than those in Saito's category $D(\mathscr{O}_X, \mathrm{Diff}_X)$. In particular in $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ Herrera–Lieberman defined an internal tensor functor and an internal Hom functor (using the graded ring $\Omega_X^\bullet$) and also a direct and an inverse image.

In the filtered case P. Du Bois studied the Grothendieck operators in his work [5] but he claimed that the formalism is only partial. We point out that in the derived category of holonomic $\mathscr{D}_X$-modules Mebkhout provided a complete formalism of six Grothendieck operators in his work [9] and so, translating that in the Du Bois category we know that there exists a suitable subcategory which is stable for these operators.

This work is devoted to the comparisons between the categories of differential complexes $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$ and $D(\mathscr{O}_X, \mathrm{Diff}_X)$.

In the first section we recall some general definitions we need in this paper and the notations we will use. In Section 2 we compare Saito category $D(\mathscr{O}_X, \mathrm{Diff}_X)$ with $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$. In fact we prove that the canonical functor between the localized categories $\lambda_1 : D_1(\mathscr{O}_X, \mathrm{Diff}_X) \to D(\mathscr{O}_X, \mathrm{Diff}_X)$ is an equivalence of categories. In Section 3 we extend this comparison result to the filtered case proving that the filtered Du Bois category $DF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ and Saito's $DF(\mathscr{O}_X, \mathrm{Diff}_X)$ are equivalent, thus answering a question posed by Saito in [10, 2.2.11].

The formalism of six Grothendieck operators in $D_1^b(\mathscr{O}_X, \mathrm{Diff}_X)$ and some applications to the notion of regular holonomic $\mathscr{D}_X$-modules are the content of a work in progress.

1. "Notations" and definitions

1.1. Principal parts sheaves and ring of differential operators

In this paper we consider X a smooth scheme of finite type over a field A of characteristic 0, or a smooth analytic variety (so over $\mathbb{C}$). We will denote by $\mathscr{P}_X^{\cdot} := \{\mathscr{P}_X^m\}_{\mathbb{Z}}$ the projective system of sheaves of principal parts [6, IV.16.8], by $q_{m,n} : \mathscr{P}_X^m \to \mathscr{P}_X^n$ the usual epimorphisms and by $q_m = q_{m,0} : \mathscr{P}_X^m \to \mathscr{O}_X$ the map induced by the diagonal embedding $X \to X \times X$. Every $\mathscr{P}_X^m$ has two canonical structures of $\mathscr{O}_X$-module induced by the projections $\pi_i : X \times X \to X$ $(i = 1, 2)$. By notation every time we form a tensor product with $\mathscr{P}_X^m$, we use the structure nearest to the tensor product (q_m is $\mathscr{O}_X$-linear for both these structures).

The $\mathscr{O}_X$ dual of $\mathscr{P}_X^{\cdot}$ with its "left" $\mathscr{O}_X$-structure is the inductive system $\mathscr{D}_{X,\cdot} := \mathscr{H}om_{\mathscr{O}_X}(\mathscr{P}_X^{\cdot}, \mathscr{O}_X)$ where $i_{m,n} : \mathscr{D}_{X,m} \to \mathscr{D}_{X,n}$ (with $m \leqslant n$) are the inclusions dual to $q_{n,m}$ and $i_m := i_{0,m} : \mathscr{O}_X \to \mathscr{D}_{X,m}$ is the inclusion dual to q_m.

The canonical morphisms $\delta^{m,n} : \mathscr{P}_X^{m+n} \to \mathscr{P}_X^m \otimes_{\mathscr{O}_X} \mathscr{P}_X^n$ [6, IV.16.8.9.1] define a co-ring structure in the projective system that is the canonical stratification on $\mathscr{P}_X^{\cdot}$. By duality they induce the maps $m_{m,n} : \mathscr{D}_{X,m} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,n} \to \mathscr{D}_{X,m+n}$ which provide the Ind-object $\mathscr{D}_{X,\cdot}$ of a ring structure (see [6, I.0.1.6] for the definition of algebraic structures on a category). This ring structure induces the ring structure on the inductive limit $\mathscr{D}_X = \varinjlim_{m \in \mathbb{Z}} \mathscr{D}_{X,m}$ called the ring of differential operators.

Any $\mathscr{D}_{X,m}$ has two structures of $\mathscr{O}_X$-modules induced by left (respectively right) multiplication by elements of $\mathscr{O}_X$. In the tensor product we always use the nearest structure.

We denote by $\Omega_X^\bullet$ the De Rham complex of differential forms and by $\Theta_X^{-i} := \mathscr{H}om_{\mathscr{O}_X}(\Omega_X^i, \mathscr{O}_X)$ the tangent complex. Moreover let $d := d_X$ be the dimension of X; we denote by $\omega_X := \Omega_X^d$ the sheaf of differential forms of maximum degree.

1.2. Differential operators

[6, IV.16.8] Given $\mathscr{F}$ and $\mathscr{G}$ two $\mathscr{O}_X$-modules, a differential operator of order at most n is a morphism $f : \mathscr{F} \to \mathscr{G}$ which could be factorized as

$$\begin{tikzcd}
 & \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,n} \arrow[d, "\mathrm{id}_{\mathscr{G}} \otimes p_n"] \\
\mathscr{F} \arrow[ur, "\overline{f}"] \arrow[r, "f"] \arrow[d, "d_n \otimes \mathrm{id}_{\mathscr{F}}"'] & \mathscr{G} \\
\mathscr{P}_X^n \otimes_{\mathscr{O}_X} \mathscr{F} \arrow[ur, "\widetilde{f}"'] &
\end{tikzcd}$$

where $d_n : \mathscr{O}_X \to \mathscr{P}_X^n$ is the universal differential operator of order n induced by π_2, so locally d_n sends $x \mapsto 1 \otimes x$ (respectively p_n is the projection dual to d_n which locally sends a differential operator of $\mathscr{O}_X$ into its value in 1) and $\widetilde{f}$ (respectively $\overline{f}$) is an $\mathscr{O}_X$-linear morphism where the $\mathscr{O}_X$-module structure on $\mathscr{P}_X^n \otimes_{\mathscr{O}_X} \mathscr{F}$ is given by the π_1-structure of $\mathscr{P}_X^n$ (respectively where the $\mathscr{O}_X$-module structure on $\mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,n}$ is given by the right multiplication on $\mathscr{D}_{X,n}$).

1.3. Definition *(Herrera–Lieberman differential complexes).* As in [7, §2] or [1, II.5], the category $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ is defined as the category of complexes of differential operators of order at most one, that is:

(i) objects in $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ are complexes whose terms are $\mathscr{O}_X$-modules and whose differentials are differential operators of order at most one;
(ii) morphisms between such complexes are morphisms of complexes which are $\mathscr{O}_X$-linear maps.

We denote by $C_1^*(\mathscr{O}_X, \mathrm{Diff}_X)$ with $* \in \{b, -, +\}$ the full subcategory of $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ whose objects are bounded, bounded above, bounded below complexes.

1.4. Proposition. *The category $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ is equivalent to the category of graded left $\mathscr{C}_X^\bullet$-modules where $\mathscr{C}_X^\bullet \cong \Omega_X^{\bullet-1} D \oplus \Omega_X^\bullet$ is the "mapping cylinder" of the identity map of $\Omega_X^\bullet$. It is a graded $\mathscr{O}_X$-algebra, whose product is defined using the wedge product of $\Omega_X^\bullet$ and $D^2 = 0$, while the structure of complex is defined by $D\alpha = (d\alpha_1 + (-1)^i \alpha_2)D + d\alpha_2$ if $\alpha = \alpha_1 D + \alpha_2$ with $\alpha_1 \in \Omega_X^{i-1}$ and $\alpha_2 \in \Omega_X^i$. Therefore, the category $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ has enough injectives* [7, §2].

1.5. Remark. We can regard each $\mathscr{C}_X^\bullet$-module as an $\Omega_X^\bullet$-one.

In particular $\Omega_X^\bullet$ is the DG (differential graded) algebra of differential forms.

So Herrera and Lieberman proved that the bi-functors $\_ \otimes_{\Omega_X^\bullet} \_$ and $\mathscr{H}om_{\Omega_X^\bullet}(\_,\_)$ are functors in $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ which are adjoint. Moreover given $f : X \to Y$ a map of schemes over A as in our setting, they define the functors $f_*(\_)$ (which is the usual direct image as abelian sheaves) and its left adjoint $f^*(\_) := \mathscr{C}_X^\bullet \otimes_{f^{-1}(\mathscr{C}_Y^\bullet)} f^{-1}(\_) = \Omega_X^\bullet \otimes_{f^{-1}(\Omega_Y^\bullet)} f^{-1}(\_)$.

1.6. Definition. A homotopy between two morphisms in $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ is a homotopy in the sense of the category of complexes of abelian sheaves, except that the homotopy operator (of degree -1) is taken to be $\mathscr{O}_X$-linear (see [7, §2]). We denote by $K_1(\mathscr{O}_X, \mathrm{Diff}_X)$ the category $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ up to null-homotopic maps.

1.7. Remark. The category $K_1(\mathscr{O}_X, \mathrm{Diff}_X)$ is triangulated where the endomorphism is the usual shift functor and the mapping cone is that in $K(A_X)$ (category of A_X-vector spaces).

1.8. Definition. In [11] Saito defines a differential operator, not even of finite type, as a A_X-linear morphism $d : \mathscr{F} \to \mathscr{G}$, between two $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, which could be factorized (in a unique way) as

$$\begin{tikzcd} \mathscr{F} \arrow[r, "d"] \arrow[dr, "\bar{d}"'] & \mathscr{G} \\ & \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_X \arrow[u, "\mathrm{id}_{\mathscr{G}} \otimes_{\mathscr{O}_X} p"'] \end{tikzcd}$$

where $\bar{d}$ is an $\mathscr{O}_X$-linear map, $p = \varinjlim_n p_n$ and $\mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ is an $\mathscr{O}_X$-module for the right multiplication of $\mathscr{D}_X$.

We note that the morphism $\bar{d}$ induces, by extension of scalars, a morphism of right $\mathscr{D}_X$-modules $\bar{d}' : \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{D}_X$.

We use the notation $M(\mathscr{O}_X, \mathrm{Diff}_X)$ for the additive category whose objects are $\mathscr{O}_X$-modules and morphisms are differential operators between them.

1.9. Definition *(Saito differential complexes).* In [11] Saito defines the equivalence of categories

$$\begin{aligned} \mathrm{DR}_X^{-1} : M(\mathscr{O}_X, \mathrm{Diff}_X) &\longrightarrow M_i(\mathscr{D}_X)^r \\ \mathscr{F} &\longmapsto \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X \\ d &\longmapsto \bar{d}' \end{aligned}$$

where $M_i(\mathscr{D}_X)^r$ is the full subcategory of right $\mathscr{D}_X$-modules whose objects are induced modules (i.e. they are isomorphic to $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ for some $\mathscr{O}_X$-module $\mathscr{F}$).

Let $K(\mathscr{O}_X, \mathrm{Diff}_X)$ be the category of complexes in $M(\mathscr{O}_X, \mathrm{Diff}_X)$ up to homotopy operators. Then the DR_X^{-1} functor extends to a triangulated functor $\mathrm{DR}_X^{-1} : K(\mathscr{O}_X, \mathrm{Diff}_X) \to K_i(\mathscr{D}_X)^r \to K(\mathscr{D}_X)^r$. We denote by $K^*(\mathscr{O}_X, \mathrm{Diff}_X)$ with $* \in \{b, -, +\}$ the full subcategory of $K(\mathscr{O}_X, \mathrm{Diff}_X)$ whose objects are bounded, bounded above, bounded below complexes.

1.10. Definition. We define the category $D(\mathscr{O}_X, \mathrm{Diff}_X)$ (respectively $D^*(\mathscr{O}_X, \mathrm{Diff}_X)$ with $* \in \{b, +, -\}$) by localizing the category $K(\mathscr{O}_X, \mathrm{Diff}_X)$ (respectively $K^*(\mathscr{O}_X, \mathrm{Diff}_X)$) with respect to the multiplicative system

$$\Sigma_{\mathrm{DR}_X^{-1}} := \left\{ f \in K(\mathscr{O}_X, \mathrm{Diff}_X) \;\middle|\; \mathrm{DR}_X^{-1}(f) \text{ is a quasi-isomorphism in } D(\mathscr{D}_X)^r \right\}.$$

The DR^{-1}-quasi-isomorphisms are called D-quasi-isomorphism (or D-qis). The category $D(\mathscr{O}_X, \mathrm{Diff}_X)$ is a triangulated category.

We recall that Saito proved in [11] that the usual De Rham functor for right $\mathscr{D}_X$-modules (see [3])

$$\mathrm{DR}_X := - \otimes^{\mathbf{L}}_{\mathscr{D}_X} \mathscr{O}_X : D^b(\mathscr{D}_X)^r \longrightarrow D^b(A_X)$$

factors as

$$\begin{tikzcd} D^b(\mathscr{D}_X)^r \arrow[r, "\widetilde{\mathrm{DR}}_X"] \arrow[dr, "\mathrm{DR}_X"'] & D^b(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[d] \\ & D^b(A_X) \end{tikzcd}$$

where the vertical arrow is the functor obtained by forgetting all structure of an object in $D^b(\mathscr{O}_X, \mathrm{Diff}_X)$ but that of A_X-module; and $\widetilde{\mathrm{DR}}_X$ is the functor

$$\begin{aligned} \widetilde{\mathrm{DR}}_X : D^b(\mathscr{D}_X)^r &\longrightarrow D^b(\mathscr{O}_X, \mathrm{Diff}_X), \\ \mathscr{M}^\bullet &\longmapsto \mathscr{M}^\bullet \otimes^\bullet_{\mathscr{O}_X} \Theta^\bullet_X. \end{aligned} \tag{1.10.1}$$

$\Theta_X^i = \bigwedge^{-i} \Theta_X$ which could be extended to unbounded complexes. In fact the Spencer complex $Sp^\bullet(\mathscr{O}_X)$

$$\begin{array}{ccccccccc} & -2 & & -1 & & 0 & & 1 & \\ \cdots \longrightarrow & \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^2 \Theta_X & \longrightarrow & \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta_X & \longrightarrow & \mathscr{O}_X & \longrightarrow & 0 & \cdots \end{array} \tag{1.10.2}$$

is a resolution of $\mathscr{O}_X$ (via $p : \mathscr{D}_X \to \mathscr{O}_X$) by locally free (so $\_ \otimes_{\mathscr{D}_X} \_$-acyclic) left $\mathscr{D}_X$-modules. Then $\mathscr{M}^\bullet \otimes^\bullet_{\mathscr{D}_X} Sp^\bullet(\mathscr{O}_X) \xrightarrow{\cong} \mathscr{M}^\bullet \otimes^{\mathbf{L}}_{\mathscr{D}_X} \mathscr{O}_X$. We recall that in a local system of coordinates $x_1, \ldots, x_d$ the differentials of the Spencer complex are defined (in the generators) as:

$$\begin{array}{ccc} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta_X^{-i} & \xrightarrow{d^i_{Sp\,\mathscr{O}_X}} & \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta_X^{-i+1} \\ P \otimes \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_i}} & \longmapsto & \sum_k (-1)^{k+1} P \frac{\partial}{\partial x_{j_k}} \otimes \frac{\partial}{\partial x_{j_1}} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial x_{j_k}}} \wedge \cdots \wedge \frac{\partial}{\partial x_{j_i}}. \end{array} \tag{1.10.3}$$

So $\mathscr{M}^\bullet \otimes^{\mathbf{L}}_{\mathscr{D}_X} \mathscr{O}_X \cong \mathscr{M}^\bullet \otimes^\bullet_{\mathscr{D}_X} (\mathscr{D}_X \otimes^\bullet_{\mathscr{O}_X} \Theta_X^\bullet) \cong \mathscr{M}^\bullet \otimes^\bullet_{\mathscr{O}_X} \Theta_X^\bullet$ where the last map is an isomorphism of graded objects and it is taken as definition for the differentials of the complex $\mathscr{M}^\bullet \otimes^\bullet_{\mathscr{O}_X} \Theta_X^\bullet$. By Saito Theorem [11, 1.10] the functor $\widetilde{\mathrm{DR}}_X$ preserves D-quasi-isomorphisms.

Moreover we recall that for $\mathscr{M}^\bullet \in D^b(\mathscr{D}_X)^r$ we have the following three descriptions of $\mathrm{DR}_X(\mathscr{M}^\bullet)$:

$$\begin{aligned} \mathrm{DR}_X(\mathscr{M}^\bullet) &= \mathscr{M}^\bullet \otimes^{\mathbf{L}}_{\mathscr{D}_X} \mathscr{O}_X \\ &= \mathscr{M}^\bullet \otimes^\bullet_{\mathscr{O}_X} \Theta_X^\bullet \\ &= \mathbf{R}\mathscr{H}om_{\mathscr{D}_X}(\omega_X, \mathscr{M}^\bullet)[d] \end{aligned}$$

in $D^b(A_X)$ where $d = \dim X$.

1.11. Theorem. *(See [11, Theorem 1.10].) The functors*

$$\begin{aligned} \widetilde{\mathrm{DR}}_X : D(\mathscr{D}_X)^r &\longrightarrow D(\mathscr{O}_X, \mathrm{Diff}_X) \\ \mathscr{M}^\bullet &\longmapsto \mathscr{M}^\bullet \otimes^\bullet_{\mathscr{O}_X} \Theta_X^\bullet \end{aligned}$$

and

$$\begin{aligned} \mathrm{DR}_X^{-1} : D(\mathscr{O}_X, \mathrm{Diff}_X) &\longrightarrow D(\mathscr{D}_X)^r \\ \mathscr{F}^\bullet &\longmapsto \mathscr{F}^\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X \end{aligned}$$

are equivalences of triangulated categories one inverse to each other.

1.12. Remark. In Definition 1.1 we have introduced the Herrera–Lieberman category of complexes $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$. The main difference between $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ and $C(\mathscr{O}_X, \mathrm{Diff}_X)$ is that Herrera–Lieberman allow only differential operators of order at most one and morphisms between complexes are $\mathscr{O}_X$-linear, while Saito considers complexes with arbitrary differential operators and also morphisms between complexes given by differential operators. We observe that the morphism $p:\mathscr{D}_X \to \mathscr{O}_X$ is a differential operator (for the right multiplication structure on $\mathscr{D}_X$) but it is not of finite order.

There is a natural functor $\lambda_1 : C_1(\mathscr{O}_X, \mathrm{Diff}_X) \to C(\mathscr{O}_X, \mathrm{Diff}_X)$ which sends objects of the first category into themselves regarded as objects in $C(\mathscr{O}_X, \mathrm{Diff}_X)$. This functor is not faithful.

1.13. Definition. We define $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$ to be the category obtained localizing the category $K_1(\mathscr{O}_X, \mathrm{Diff}_X)$ of Herrera and Lieberman with respect to the multiplicative system of morphisms

$$\Sigma_{1,\mathrm{DR}_X^{-1}} := \left\{ f \in K_1(\mathscr{O}_X, \mathrm{Diff}_X) \;\middle|\; \mathrm{DR}_X^{-1} \circ \lambda_1(f) \text{ is a quasi-isomorphism in } D(\mathscr{D}_X)^r \right\}$$

called the system of D-quasi-isomorphisms. We refer to this category as the Herrera–Lieberman localized category.

1.14. Remark. We note that the functor λ_1 respects D-quasi-isomorphisms so it defines a localized triangulated functor (which we again denote by λ_1)

$$\lambda_1 : D_1(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow D(\mathscr{O}_X, \mathrm{Diff}_X).$$

It seems to be not straightforward to prove that this functor is fully faithful.

We observe that the functor $\widetilde{\mathrm{DR}}_X$, extended to unbounded complexes, also factors through $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$. In fact for any object $\mathscr{M}^\bullet \in D(\mathscr{D}_X)^r$, $\mathscr{M}^\bullet \otimes^\bullet_{\mathscr{O}_X} \Theta^\bullet_X$ belongs to $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$ and the same holds true for morphisms. So we obtain the commutative diagram

$$\begin{tikzcd}
D(\mathscr{D}_X)^r \arrow[r, "\widetilde{\mathrm{DR}}_{1,X}"] \arrow[dr, "\widetilde{\mathrm{DR}}_X"'] & D_1(\mathscr{O}_X, \mathrm{Diff}_X) \arrow[d, "\lambda_1"] \\
 & D(\mathscr{O}_X, \mathrm{Diff}_X).
\end{tikzcd}$$

The composition $\widetilde{\mathrm{DR}}_{1,X}\mathrm{DR}_X^{-1}$ defines a triangulated functor from $D(\mathscr{O}_X, \mathrm{Diff}_X)$ to $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$. In the sequel, we will prove that λ_1 and $\widetilde{\mathrm{DR}}_{1,X}\mathrm{DR}_X^{-1}$ are quasi-inverses of each other and so define an equivalence of triangulated categories.

We denote $\mathrm{DR}_{1,X}^{-1} = \mathrm{DR}_X^{-1} \circ \lambda_1$ which would be a quasi-inverse of $\widetilde{\mathrm{DR}}_{1,X}$.

2. Comparison between Saito and HL-localizations

2.1. Remark. Let $\mathscr{F}^\bullet$ be an object in $C_1(\mathscr{O}_X, \mathrm{Diff}_X)$. By definition the differential $d^i_{\mathscr{F}} : \mathscr{F}^i \to \mathscr{F}^{i+1}$ is a differential operator of order at most one. So it defines in a unique way a morphism $\bar{d}^i_{\mathscr{F}} : \mathscr{F}^i \to \mathscr{F}^{i+1} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,1}$.

This morphism extends in a unique way to a morphism of $\mathscr{D}_X$-modules $\mathrm{DR}_X^{-1} d^i_{\mathscr{F}} : \mathscr{F}^i \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{F}^{i+1} \otimes_{\mathscr{O}_X} \mathscr{D}_X$. Locally for each section s of $\mathscr{F}^i$, $\bar{d}^i_{\mathscr{F}}(s)$ is a section of $\mathscr{F}^{i+1} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,1}$ so it may be locally written in a unique way as

$$\bar{d}^i_{\mathscr{F}}(s) = d^i_{\mathscr{F}}(s) \otimes 1 + \sum_{j=1}^{d} d^i_{x_j}(s) \otimes \frac{\partial}{\partial x_j} \tag{2.1.1}$$

using the $\mathscr{O}_X$-base of $\mathscr{D}_{X,1}$ given in local coordinates by $1, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}$. The maps $d^i_{\mathscr{F}} : \mathscr{F}^i \to \mathscr{F}^{i+1}$ are the differentials of the complex $\mathscr{F}^\bullet$; while $d^i_{x_j} : \mathscr{F}^i \to \mathscr{F}^{i+1}$ are maps of $\mathscr{O}_X$-modules depending on the choice of the coordinates.

2.2. Definition. Let by definition $\sigma^{i,j}_{\mathscr{F}} : \mathscr{F}^i \to \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \Theta_X^{-j}$ be the $\mathscr{O}_X$-linear maps defined as follow:

$$\begin{tikzcd}
\mathscr{F}^i \arrow[r, "\bar{d}"] \arrow[drrr, "\sigma^{i,j}_{\mathscr{F}}"'] & \mathscr{F}^{i+1} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,1} \arrow[r, "\bar{d}\otimes\mathrm{id}"] & \cdots \arrow[r] & \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,1} \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} \mathscr{D}_{X,1} \arrow[d] \\
 & & & \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \Theta_X^{-j}.
\end{tikzcd}$$

The vertical map is the identity on $\mathscr{F}^{i+1}$ tensor the map obtained by the composition of the projections $\mathscr{D}_{X,1} \to \Theta_X^1$ and $\Theta_X^1 \otimes_{\mathscr{O}_X} \cdots \otimes_{\mathscr{O}_X} \Theta_X^1 \to \Theta_X^{-j}$. We observe that these maps $\sigma^{i,j}_{\mathscr{F}}$ are related to the structural morphisms of the $\Omega_X^\bullet$-module $\mathscr{F}^\bullet$

$$\mathscr{F}^i \otimes_{\mathscr{O}_X} \Omega_X^j \longrightarrow \mathscr{F}^{i+j};$$

in fact they are adjoints because $\Theta_X^{-j} := \bigwedge^j \Theta_X \cong \mathscr{H}om_{\mathscr{O}_X}(\Omega_X^j, \mathscr{O}_X)$ and

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{O}_X}\big(\mathscr{F}^i \otimes_{\mathscr{O}_X} \Omega_X^i, \mathscr{F}^{i+j}\big) &\cong \mathrm{Hom}_{\mathscr{O}_X}\big(\mathscr{F}^i, \mathscr{H}om_{\mathscr{O}_X}\big(\Omega_X^i, \mathscr{F}^{i+j}\big)\big) \\
&\cong \mathrm{Hom}_{\mathscr{O}_X}\big(\mathscr{F}^i, \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \Theta_X^{-j}\big)
\end{aligned}$$

where the last isomorphism holds true because Ω_X^i is locally free of finite rank.

Now we prove a technical lemma which would be used in the proof of our main theorem.

2.3. Lemma. *Given $\mathscr{F}^\bullet \in C_1(\mathscr{O}_X, \mathrm{Diff}_X)$; the morphisms $d^i_{\mathscr{F}}$, $d^i_{x_j}$ of* (2.1.1) *for $i \in \mathbb{Z}$ and $j \in \{0, \dots, d\}$ satisfy the following conditions*:

(i) $d^{i+1}_{\mathscr{F}} \circ d^i_{\mathscr{F}} = 0$,
(ii) $d^{i+1}_{x_j} \circ d^i_{\mathscr{F}} + d^{i+1}_{\mathscr{F}} \circ d^i_{x_j} = 0$,
(iii) $d^{i+1}_{x_j} \circ d^i_{x_k} + d^{i+1}_{x_k} \circ d^i_{x_j} = 0$,
(iv) $d^{i+1}_{x_j} \circ d^i_{x_j} = 0$.

Proof. The first condition is given by the hypothesis $\mathscr{F}^\bullet \in C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ so the composition $d_{\mathscr{F}}^{i+1} \circ d_{\mathscr{F}}^i = 0$. The conditions (ii) to (iv) follow from the condition that the composition

$$\overline{d_{\mathscr{F}}^{i+1} \circ d_{\mathscr{F}}^i} : \mathscr{F}^i \longrightarrow \mathscr{F}^{i+2} \otimes_{\mathscr{O}_X} \mathscr{D}_{X,2}$$

is zero because it corresponds to $d_{\mathscr{F}}^{i+1} \circ d_{\mathscr{F}}^i = 0$. In fact the morphism $\overline{d}^i_{\mathscr{F}}$ associated to $d^i_{\mathscr{F}}$ is induced by the functor DR_X^{-1}, and the latter is compatible with the composition of morphisms. $\square$

2.4. Definition. Let $\mathscr{F}^\bullet \in C_1(\mathscr{O}_X, \mathrm{Diff}_X)$. We define for each $i \in \mathbb{Z}$ and $j \in \{0, \dots, d\}$ ($d = \dim X$) the maps

$$\eta_{\mathscr{F}}^{i,0} = \mathrm{id}_{\mathscr{F}^i} : \mathscr{F}^i \longrightarrow \mathscr{F}^i \tag{2.4.1}$$

and

$$\begin{aligned} \eta_{\mathscr{F}}^{i,j} : \mathscr{F}^i &\longrightarrow \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \Theta_X^{-j} \\ s &\longmapsto \sum_{i_1 < \dots < i_j} d_{x_{i_j}}^{i+j-1} \circ \dots \circ d_{x_{i_1}}^i (s) \otimes \frac{\partial}{\partial x_{i_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{i_j}} \end{aligned} \tag{2.4.2}$$

for $j \in \{1, \dots, d\}$. Then (up to a sign $(-1)^{\binom{j}{2}}$) we have that $\sigma_{\mathscr{F}}^{i,j} = j! \eta_{\mathscr{F}}^{i,j}$, so these maps $\eta_{\mathscr{F}}^{i,j}$ do not depend on local coordinates.

2.5. Definition. Let $i : \mathscr{O}_X \to \mathscr{D}_X$ be the usual inclusion which is linear for both the $\mathscr{O}_X$-module structures of $\mathscr{D}_X$. Given $\mathscr{F}^\bullet \in C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ we define the morphisms

$$\Phi_{\mathscr{F}}^i : \mathscr{F}^i \longrightarrow \bigoplus_{j=0}^{d} \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta_X^{-j}$$

for each $i \in \mathbb{Z}$ in the following way: we consider the composition

$$\begin{tikzcd} \mathscr{F}^i \arrow[r, "\eta_{\mathscr{F}}^{i,j}"] \arrow[dr, "\Phi_{\mathscr{F}}^{i,j}"'] & \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \Theta_X^{-j} \arrow[d, "\mathrm{id}_{\mathscr{F}^{i+j}} \otimes i \otimes \mathrm{id}_{\Theta_X^{-j}}"] \\ & \mathscr{F}^{i+j} \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta_X^{-j} \end{tikzcd}$$

and by definition $\Phi_{\mathscr{F}}^i := \sum_{j=0}^d \Phi_{\mathscr{F}}^{i,j}$.

We want to prove that the morphisms $\Phi_{\mathscr{F}}^i : \mathscr{F}^i \to (\widetilde{\mathrm{DR}}_X \mathrm{DR}_X^{-1}(\mathscr{F}^\bullet))^i$ define a morphism of complexes.

2.6. Theorem. *The maps $\Phi^i_{\mathscr{F}}$ of* 2.5 *define*

$$\Phi_{\mathscr{F}}:\mathscr{F}^\bullet \longrightarrow \widetilde{\mathrm{DR}}_X {\mathrm{DR}_X}^{-1}(\mathscr{F}^\bullet)$$

which is a morphism of complexes in $C_1(\mathscr{O}_X,\mathrm{Diff}_X)$.

Proof. We have to prove that the diagram

$$\begin{tikzcd}
\mathscr{F}^i \arrow[rr, "d^i_{\mathscr{F}}"] \arrow[d, "\Phi^i_{\mathscr{F}}"] & & \mathscr{F}^{i+1} \arrow[d, "\Phi^{i+1}_{\mathscr{F}}"] \\
\bigoplus_{j=0}^d \mathscr{F}^{i+j}\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^{-j} \arrow[rr, "d^i_{\widetilde{\mathrm{DR}}\mathrm{DR}^{-1}}"'] & & \bigoplus_{j=0}^d \mathscr{F}^{i+1+j}\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^{-j}
\end{tikzcd}$$

is commutative.

We recall that $\widetilde{\mathrm{DR}}_X {\mathrm{DR}_X}^{-1}(\mathscr{F}^\bullet)=(\mathscr{G}^{\bullet\bullet})_{\mathrm{tot}}$ where

$$\mathscr{G}^{p,q}=\mathscr{F}^q\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^p$$

and

$$d'^{p,q}_{\mathscr{G}}:\mathscr{F}^q\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^p\longrightarrow\mathscr{F}^q\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^{p+1}$$

is

$$d'^{p,q}_{\mathscr{G}}=\mathrm{id}_{\mathscr{F}^q}\otimes\bigl(-d^p_{Sp\mathscr{O}_X}\bigr); \tag{2.6.1}$$

while

$$d''^{p,q}_{\mathscr{G}}:\mathscr{F}^q\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^p\longrightarrow\mathscr{F}^{q+1}\otimes_{\mathscr{O}_X}\mathscr{D}_X\otimes_{\mathscr{O}_X}\Theta_X^p$$

is

$$d''^{p,q}_{\mathscr{G}}={\mathrm{DR}_X}^{-1}\bigl(d^q_{\mathscr{F}}\bigr)\otimes\mathrm{id}_{\Theta_X^p}, \tag{2.6.2}$$

where $d^p_{Sp\mathscr{O}_X}$ was defined in (1.10.3).

We consider an additive category $\mathscr{A}$, the category of complexes and the category of naïf bounded bicomplexes on it (naïf means that the differentials commute as in [6, III.0.11.3.1], and bounded means that in any anti-diagonal only a finite number of terms are not isomorphic to zero). We recall that given $I^{\bullet,\bullet}$ a bicomplex with commuting differentials $d'^{p,q}_I:I^{p,q}\to I^{p+1,q}$ and $d''^{p,q}_I:I^{p,q}\to I^{p,q+1}$, the total complex associated to it is denoted by $I^\bullet_{\mathrm{tot}}$ with $I^r_{\mathrm{tot}}:=\bigoplus_{p+q=r}I^{p,q}$ and $d_{I_{\mathrm{tot}}}(x)=d'^{p,q}_I(x)+(-1)^p d''^{p,q}_I(x)$ for any $x\in I^{p,q}$. We want now to describe the morphisms of complexes between a complex and the total complex associated to a bicomplex.

The set $\mathrm{Hom}(A^\bullet,B^\bullet_{\mathrm{tot}})$ of morphisms of complexes between a complex and the total complex of a bicomplex is the set of families of maps $\{\varphi^a:A^a\to B^a_{\mathrm{tot}}\}_a$ such that $d^a_{B_{\mathrm{tot}}}\circ\varphi^a=$

$\varphi^{a+1} \circ d_A^a$. Then $\mathrm{Hom}(A^\bullet, B^\bullet_{\mathrm{tot}})$ is isomorphic to the set of families of maps $\{\varphi^{a,p} : A^a \to B^{p,a-p}\}_{a,p}$ satisfying the following conditions

$$\varphi^{a+1,p} d_A^a - (-1)^p d_B''^{p,a-p} \varphi^{a,p} = d_B'^{p-1,a-p+1} \varphi^{a,p-1} \tag{2.6.3}$$

for any a, p.

So we have only to prove that

$$\Phi_{\mathscr{F}}^{a+1,p} \circ d_{\mathscr{F}}^a - (-1)^p d_{\mathscr{G}}''^{-p,a+p} \circ \Phi_{\mathscr{F}}^{a,p} = d_{\mathscr{G}}'^{-p-1,a+p+1} \circ \Phi_{\mathscr{F}}^{a,p+1} \tag{2.6.4}$$

is true.

It is enough to check these relations locally, choosing local coordinates $x_1, \dots, x_n$. Let s be a section of $\mathscr{F}^a$, then

$$\Phi_{\mathscr{F}}^{a+1,p} \circ d_{\mathscr{F}}^a(s) = \sum_{i_1 < \cdots < i_p} d_{x_{i_p}}^{a+p} \cdots d_{x_{i_1}}^{a+1} d_{\mathscr{F}}^a(s) \otimes 1 \otimes \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}};$$

while

$$\begin{aligned}
&d_{\mathscr{G}}''^{-p,a+p} \circ \Phi_{\mathscr{F}}^{a,p}(s) \\
&\quad = d_{\mathscr{G}}''^{-p,a+p} \biggl(\sum_{i_1 < \cdots < i_p} d_{x_{i_p}}^{a+p-1} \cdots d_{x_{i_1}}^a(s) \otimes 1 \otimes \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}} \biggr) \\
&\quad = \sum_{i_1 < \cdots < i_p} d_{\mathscr{F}}^{a+p} \bigl(d_{x_{i_p}}^{a+p-1} \cdots d_{x_{i_1}}^a(s) \bigr) \otimes 1 \otimes \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}} \\
&\qquad + \sum_k \sum_{i_1 < \cdots < i_p} d_{x_k}^{a+p} \bigl(d_{x_{i_p}}^{a+p-1} \cdots d_{x_{i_1}}^a(s) \bigr) \otimes \frac{\partial}{\partial x_k} \otimes \biggl(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}} \biggr) \\
&\quad = \sum_{i_1 < \cdots < i_p} (-1)^p d_{x_{i_p}}^{a+p} \cdots d_{x_{i_1}}^{a+1} d_{\mathscr{F}}^a(s) \otimes 1 \otimes \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_p}} \\
&\qquad + \sum_{i_1 < \cdots < i_{p+1}} \sum_l (-1)^{p+l+1} d_{x_{i_{p+1}}}^{a+p} \cdots d_{x_{i_1}}^a(s) \otimes \frac{\partial}{\partial x_{i_l}} \otimes \biggl(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial x_{i_l}}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{p+1}}} \biggr).
\end{aligned}$$

On the right-hand side of (2.6.4) we obtain

$$\begin{aligned}
&d_{\mathscr{G}}'^{-p-1,a+p+1} \circ \Phi_{\mathscr{F}}^{a,p+1}(s) \\
&\quad = d_{\mathscr{G}}'^{-p-1,a+p+1} \biggl(\sum_{i_1 < \cdots < i_{p+1}} d_{x_{i_{p+1}}}^{a+p} \cdots d_{x_{i_1}}^a(s) \otimes 1 \otimes \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{p+1}}} \biggr) \\
&\quad = \sum_{i_1 < \cdots < i_{p+1}} \sum_l (-1)^l d_{x_{i_{p+1}}}^{a+p} \cdots d_{x_{i_1}}^a(s) \otimes \frac{\partial}{\partial x_{i_l}} \otimes \biggl(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \widehat{\frac{\partial}{\partial x_{i_l}}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{p+1}}} \biggr).
\end{aligned}$$

Thus we have established our assertion. □

2.7. Theorem. *The functor*

$$\lambda_1 : D_1(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow D(\mathscr{O}_X, \mathrm{Diff}_X)$$

is an equivalence of categories with quasi-inverse the functor

$$\widetilde{\mathrm{DR}}_{1,X} \circ \mathrm{DR}_X^{-1} : D(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow D_1(\mathscr{O}_X, \mathrm{Diff}_X).$$

Proof. Let $G := \widetilde{\mathrm{DR}}_{1,X} \circ \mathrm{DR}_X^{-1}$.

By Saito's results we obtain $\lambda_1 \circ G = \widetilde{\mathrm{DR}}_X \circ \mathrm{DR}_X^{-1} \xrightarrow{\cong} \mathrm{id}_{D(\mathscr{O}_X, \mathrm{Diff}_X)}$. We want to prove that there exists an isomorphism of functors $\mathrm{id}_{D_1(\mathscr{O}_X, \mathrm{Diff}_X)} \to G \circ \lambda_1$.

In 2.5 we defined a functorial morphism $\Phi^\bullet_{\mathscr{F}} : \mathscr{F}^\bullet \to G \circ \lambda_1(\mathscr{F}^\bullet)$ (for each $\mathscr{F}^\bullet \in D_1(\mathscr{O}_X, \mathrm{Diff}_X)$) which is a DR_X^{-1}-quasi-isomorphism because the triangle

$$\begin{tikzcd} \lambda_1(\mathscr{F}^\bullet) \arrow[r, "\lambda_1 \circ \Phi^\bullet_{\mathscr{F}}"] \arrow[dr, "\mathrm{id}_{\lambda_1(\mathscr{F}^\bullet)}"'] & \lambda_1 \circ G \circ \lambda_1(\mathscr{F}^\bullet) \arrow[d] \\ & \lambda_1(\mathscr{F}^\bullet) \end{tikzcd}$$

commutes (where the vertical map is that induced by the projection $\mathscr{D}_X \to \mathscr{O}_X$ and it is a quasi-isomorphism by Saito's result). So the morphism defined by the functor $\Phi : \mathrm{id}_{D_1(\mathscr{O}_X, \mathrm{Diff}_X)} \to G \circ \lambda_1$ is an isomorphism. $\square$

See also [2] for another approach to this equivalence problem.

2.8. Definition. Let $\mathscr{F}$ and $\mathscr{G}$ be two $\mathscr{O}_X$-modules. In [11, 1.11] Saito defined the set of finite differential operators $\mathrm{Hom}^f_{\mathrm{Diff}_X}(\mathscr{F}, \mathscr{G})$ and so the category $M(\mathscr{O}_X, \mathrm{Diff}_X)^f$. Then in [11, Proposition 1.2] he proved that the functors $\widetilde{\mathrm{DR}}$ and DR^{-1} induce an equivalence of categories between $D^b(\mathscr{O}_X, \mathrm{Diff}_X)^f$ and $D^b(\mathscr{D}_X)^r$. Moreover in [11, Definition 1.15] Saito defined $D^b_{\mathrm{coh}}(\mathscr{O}_X, \mathrm{Diff}_X)^f$ to be the category corresponding to $D^b_{\mathrm{coh}}(\mathscr{D}_X)^r$ (the derived category of coherent right $\mathscr{D}_X$-modules) under the previous equivalence.

2.9. Definition. An object in $D(\mathscr{D}_X)^r$ is said perfect if it is locally isomorphic to a bounded complex whose elements are induced from locally free $\mathscr{O}_X$-modules of finite type. We define $D^b_p(\mathscr{D}_X)^r$ to be the triangulated category of bounded perfect complexes. (We refer to the paper of Laumon [8, §2] for the definition of perfect complexes.) We note that in the noetherian case $D^b_p(\mathscr{D}_X)^r \cong D^b_{\mathrm{coh}}(\mathscr{D}_X)^r$. So we will use perfect complexes which are equivalent to coherent ones. Let denote by $D^b_p(\mathscr{O}_X, \mathrm{Diff}_X)$ (respectively $D^b_{1,p}(\mathscr{O}_X, \mathrm{Diff}_X)$) the full subcategory of $D^b(\mathscr{O}_X, \mathrm{Diff}_X)^f$ (respectively of $D^b_1(\mathscr{O}_X, \mathrm{Diff}_X)$) which corresponds to $D^b_p(\mathscr{D}_X)^r$ under the $\widetilde{\mathrm{DR}}$-equivalence. Notice that any object in $D^b_p(\mathscr{O}_X, \mathrm{Diff}_X)$ is locally isomorphic to a bounded complex whose elements are locally free $\mathscr{O}_X$-modules of finite type and so the differential are differential operators of finite type.

Let denote by $D^b_h(\mathscr{O}_X, \mathrm{Diff}_X)$ (respectively $D^b_{1,h}(\mathscr{O}_X, \mathrm{Diff}_X)$) the full subcategory of $D^b(\mathscr{O}_X, \mathrm{Diff}_X)^f$ (respectively of $D^b_1(\mathscr{O}_X, \mathrm{Diff}_X)$) which corresponds to $D^b_h(\mathscr{D}_X)^r$ (bounded derived category of holonomic $\mathscr{D}_X$-modules) under the $\widetilde{\mathrm{DR}}$-equivalence.

3. Comparison between Saito and Du Bois categories

3.1. Definition. By definition the category $CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ is the category whose objects are filtered complexes $(K^\bullet, d, F)$ such that:

(i) $K^\bullet$ is a complex of $\mathscr{O}_X$-modules;
(ii) F is a decreasing filtration on $K^\bullet$ given by sub-$\mathscr{O}_X$-modules and F is biregular (that is on every component K^i of $K^\bullet$, F induces a finite filtration; so there exist integers p and q such that $F^pK^i = K^i$ and $F^qK^i = 0$);
(iii) d is a relative differential operator of order at most 1 which respects the filtrations;
(iv) $\mathrm{gr}_F(d)$ is $\mathscr{O}_X$-linear.

Morphisms in $CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ are $f^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ $\mathscr{O}_X$-linear maps commuting with differentials and compatible with filtrations [4, 1].

3.2. Remark. The complex $\Omega_X^\bullet$ is filtered by truncation so $\mathrm{gr}_F^p(\Omega_X^\bullet) = \Omega_X^p$.

3.3. Definition. A filtered homotopy between $u, v : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ is a homotopy h such that $h^i(F^p\mathscr{F}^i) \subseteq F^p\mathscr{G}^{i-1}$ and h^i is $\mathscr{O}_X$-linear.

Let $KF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ be the category whose objects are those of $CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ and whose morphisms are the classes of morphisms in $CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ up to homotopy.

3.4. Definition. A morphism $f^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ is said a filtered quasi-isomorphism if $\mathrm{gr} f$ is a quasi-isomorphism where gr is the functor

$$\mathrm{gr} : CF_1(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow CG(X)$$

sending a filtered complex $(\mathscr{F}^\bullet, F)$ to its graded complex. (Here $CG(X)$ is the category of complexes of $\mathscr{O}_X$-modules with a finite graduation in each degree.)

Let $DF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ be the category obtained by localizing $KF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ with respect to filtered quasi-isomorphisms.

3.5. Remark. The categories $KF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ and $DF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ are triangulated categories where

(i) the shift functor is the usual one: $T(K)^i = K^{i+1}$, $d_{T(K)} = -d_K$ and $F_{T(K)} = F_K$;
(ii) if $f^\bullet : \mathscr{F}^\bullet \to \mathscr{G}^\bullet$ is a morphism in $CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ its mapping cone is $\mathscr{M}^\bullet := T(\mathscr{F}^\bullet) \oplus \mathscr{G}^\bullet \in C_1(\mathscr{O}_X, \mathrm{Diff}_X)$ with filtration defined by $F^p(\mathscr{M}^\bullet) = F^pT(\mathscr{F}^\bullet) \oplus F^p\mathscr{G}^\bullet$ and differential $d_{\mathscr{M}}$ defined by the matrix $\begin{bmatrix} d_{T(\mathscr{F})} & 0 \\ T(f) & d_{\mathscr{G}} \end{bmatrix}$.

3.6. Definition. Let $(\mathscr{L}_j, F)$ with $j \in \{1, 2\}$ be two filtered $\mathscr{O}_X$-modules with increasing filtration such that $F_p\mathscr{L}_j = 0$ for $p \ll 0$. By definition

$$\mathscr{H}om_{\mathrm{Diff}_X}\big((\mathscr{L}_1, F), (\mathscr{L}_2, F)\big)$$

is the sheaf of filtered differential operators, that is $\Phi \in \mathscr{H}om_{\mathrm{Diff}_X}((\mathscr{L}_1, F), (\mathscr{L}_2, F))$ if and only if the composition

$$F_p\mathscr{L}_1 \longrightarrow \mathscr{L}_1 \xrightarrow{\Phi} \mathscr{L}_2 \longrightarrow \mathscr{L}_2/F_{p-q-1}\mathscr{L}_2 \tag{3.6.1}$$

has order at most q for each p, q.

This condition implies that $\Phi(F_p\mathscr{L}_1) \subset F_p\mathscr{L}_2$ and that the map between the graded objects $\mathrm{gr}_p^F(\mathscr{L}_1) \to \mathrm{gr}_p^F(\mathscr{L}_2)$ is $\mathscr{O}_X$-linear.

3.7. Remark. We observe that if $(\mathscr{L}_1, F) \to (\mathscr{L}_2, F)$ is a filtered differential operator of order at most one then the condition (3.6.1) is equivalent to the condition given by Du Bois of having graded $\mathscr{O}_X$-linear for the objects $(\mathscr{L}_j, F')$ where $F'^p\mathscr{L}_j = F_{-p}\mathscr{L}_j$ is the opposite filtration.

3.8. Definition. We denote by $CF(\mathscr{O}_X, \mathrm{Diff}_X)$ the category of complexes of (increasing) filtered $\mathscr{O}_X$-modules and filtered differential operators.

The functor λ_1 induces a functor which we denote by

$$\lambda_1 F : CF_1(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow CF(\mathscr{O}_X, \mathrm{Diff}_X).$$

It sends objects of $CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ into themselves with the opposite filtration (which becomes increasing). We denote by $DF(\mathscr{O}_X, \mathrm{Diff}_X)$ the category obtained localizing $CF(\mathscr{O}_X, \mathrm{Diff}_X)$ with respect to filtered quasi-isomorphisms.

3.9. Theorem. *The functor*

$$\lambda_1 F : DF_1(\mathscr{O}_X, \mathrm{Diff}_X) \longrightarrow DF(\mathscr{O}_X, \mathrm{Diff}_X)$$

is an equivalence of categories; *so also is the functor*

$$\widetilde{\mathrm{DR}}_{1,X} F : DF(\mathscr{D}_X)^r \longrightarrow DF_1(\mathscr{O}_X, \mathrm{Diff}_X).$$

Proof. Saito proved that the functor $\widetilde{\mathrm{DR}}_X F : DF(\mathscr{D}_X)^r \to DF(\mathscr{O}_X, \mathrm{Diff}_X)$ is an equivalence of categories with quasi-inverse the functor $\mathrm{DR}_X^{-1} F$. We want to extend the result of the previous section to the filtered context. As for the nonfiltered case, let $GF = \widetilde{\mathrm{DR}}_{1,X} F \circ \mathrm{DR}_X^{-1} F$ we have only to prove that there is an isomorphism of functors $\mathrm{id}_{DF_1(\mathscr{O}_X, \mathrm{Diff}_X)} \to GF \circ \lambda_1 F$.

In the previous section we defined an isomorphism of functors

$$\Phi : \mathrm{id}_{D_1(\mathscr{O}_X, \mathrm{Diff}_X)} \longrightarrow \widetilde{\mathrm{DR}}_{1,X} \circ \mathrm{DR}_X^{-1} \circ \lambda_1$$

into $D_1(\mathscr{O}_X, \mathrm{Diff}_X)$. Now given $(\mathscr{F}^\bullet, F) \in CF_1(\mathscr{O}_X, \mathrm{Diff}_X)$ we want to prove that the morphism $\Phi^\bullet_{\mathscr{F}}$ respects the filtrations, so it induces an isomorphism of functors also in the filtered case. We observe that the map $\eta^{i,j}_{\mathscr{F}}$ satisfies:

$$\eta^{i,j}_{\mathscr{F}} : F_p(\mathscr{F}^i) \longrightarrow F_p(\mathscr{F}^{i+j}) \otimes_{\mathscr{O}_X} \Theta_X^{-j} \tag{3.9.1}$$

because the differentials of the complex respect the filtrations.

We recall that the filtration on the complex $\widetilde{\mathrm{DR}}_{1,X}\mathrm{DR}_X^{-1}(\mathscr{F}^\bullet)$ is built as explained in [10, 2.1.3, 2.1.5] for a single $\mathscr{O}_X$-module $\mathscr{F}$. Generalizing this construction to complexes we have that

$$F_p\left(\bigoplus_{j=0}^{d}\mathscr{F}^{i+j}\otimes\mathscr{D}_X\otimes\Theta_X^{-j}\right)=\bigoplus_{j=0}^{d}\sum_{l\geqslant 0}F_{p-l}\big(\mathscr{F}^{i+j}\big)\otimes\mathscr{D}_{X,l}\otimes\Theta_X^{-j}.$$

This implies that also $\Phi^{i,j}_{\mathscr{F}}$ respects the filtrations because $F_p(\mathscr{F}^i)$ takes image into $F_p(\mathscr{F}^{i+j})\otimes\mathscr{D}_{X,0}\otimes\Theta_X^{-j}$; so we have established our thesis. $\square$

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